ÉCOLE POLYTECHNIQUE
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On fractals, fractional splines and wavelets

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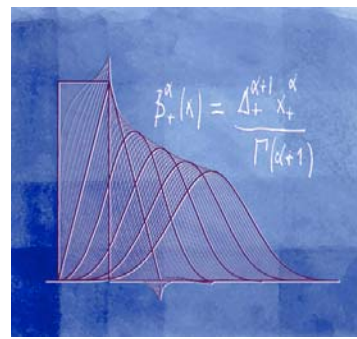

WAMA, Cargèse, July 2004

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **07 JAN 2005**	2. REPORT TYPE **N/A**	3. DATES COVERED **-**

4. TITLE AND SUBTITLE **On fractals, fractional splines and wavelets**	5a. CONTRACT NUMBER 5b. GRANT NUMBER 5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER 5e. TASK NUMBER 5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Biomedical Imaging Group EPFL, Lausanne Switzerland**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR’S ACRONYM(S) 11. SPONSOR/MONITOR’S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release, distribution unlimited

13. SUPPLEMENTARY NOTES
See also ADM001750, Wavelets and Multifractal Analysis (WAMA) Workshop held on 19-31 July 2004., The original document contains color images.

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**UU**	**48**	

FRACTALS AND PHYSIOLOGY

- Fractal characteristics:
  - Complex, patterned
  - Statistical self-similarity
  - Scale-invariant structure
  - Generated by simple iterative rules
  - $1/\omega^{2H+d}$ spectral decay

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are needed to see this picture.

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- Growth processes, biofractals

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Cardiovascular system

- Heart
  - Arterial tree
  - Dendritic anatomy

- Lung

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From Goldberger, Rigney and West

Fractal bones

- Trabecular bone

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CT of a vertebra

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µCT

Courtesy F. Peyrin ESRF

Mammograms

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DDSM: University of Florida

(Digital Database for Screening Mammography)

(Arnéodo et al., 2001)

Brain as a biofractal

(Bullmore, 1994)

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1mm

Courtesy R. Mueller ETHZ

OUTLINE

- Fractals in physiology
- Wavelets and fractals
  - Motivation for using wavelets
  - Fractal processing: order is the key
  - What about fractional differentiation
- Fractional splines
- Fractional wavelets
- Wavelets in medical imaging
  - Survey of applications
  - Analysis of functional images

Motivation for using wavelets

- Wavelets provide basis functions that are self-similar [*Mallat*, 1989]

$$\psi_{i,k} = 2^{-i/2}\psi\left(\frac{x-2^i k}{2^i}\right)$$

$$\boxed{\forall f(x) \in L_2, \quad f(x) = \sum_{i\in Z}\sum_{k\in Z} \langle f, \tilde{\psi}_{i,k}\rangle\, \psi_{i,k}(x)}$$

- Wavelets approximately decorrelate statistically self-similar processes [*Flandrin*, 1992; *Wornell*, 1993]
- Unlike Fourier exponentials, wavelets are jointly localized in space and frequency
- The basis functions themselves are fractals [*Blu-Unser*, 2002]

$\Rightarrow$ Wavelets are prime candidates for processing fractal-like signals and images

On the fractal nature of wavelets

- **Harmonic spline decomposition of wavelets**

  Theorem: Any valid compactly supported scaling function $\varphi(x)$ (or wavelet $\psi(x)$) can be expressed either as

  (1) a weighed sum of the integer shifts of a self-similar function (fractal) ;

  (2) a linear combination of harmonic splines with complex exponents.

[*Blu-Unser*, 2002]

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D4 as a sum of harmonic splines

- Sum of spline components

$$\varphi_N(x) = \sum_{n=-N/2}^{+N/2} \gamma_n s_n(x)$$

where

$$s_n(x) = \sum_{k \in Z^+} p_k (x-k)_+^{\frac{\log \lambda}{\log 2} + j\frac{2\pi n}{\log 2}}$$

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Fractal processing: order is the key !

- Vanishing moments

Classical *N*th order transform $\Leftrightarrow$ analysis wavelet $\tilde{\psi}(x)$ has *N* vanishing moments

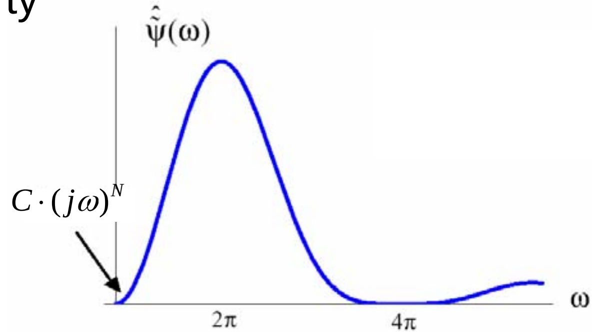
$$\int_{x \in R} x^n \tilde{\psi}(x)dx = 0, \quad n = 0,\cdots,N-1$$

$\Rightarrow$ $\tilde{\psi}$ kills all polynomials of degree $n < N$

- Multi-scale differentiation property

Property
An analysis wavelet of order *N* acts like a *N*th order differentiator:

$$\hat{\tilde{\psi}}(\omega) = O(\omega^N)$$



$\Rightarrow$ $$\langle f(x),\ \tilde{\psi}(x-u)\rangle = \frac{d^N}{du^N}\{\phi * f\}(u)$$

Smoothing kernel: $\hat{\phi}(\omega) = \hat{\tilde{\psi}}^*(\omega)/(j\omega)^N$

What about fractional differentiation ?

- Fractional differentiation operator

$$\partial^{\gamma} f(x) \quad \overset{F}{\longleftrightarrow} \quad (j\omega)^{\gamma}\hat{f}(\omega) \qquad\qquad \gamma \in R^{+}$$

- Motivation: whitening of fBM-like processes

$$\phi_f(\omega) \approx O(1/\omega^{2H+1}) \longrightarrow \boxed{\text{Fractional differentiator}} \longrightarrow \phi_w(\omega) \approx O(1)$$

White noise

$$\gamma = H + \tfrac{1}{2}$$

- QUESTION
  Are there wavelets that act like fractional differentiators ?

- ANSWER
  Not within the context of standard wavelet theory where the order is constrained to be an *integer*,
  but …

SPLINES

- Polynomial splines
- Fractional B-splines
- Properties
  - Fractional differentiation
  - Fractional order of approximation

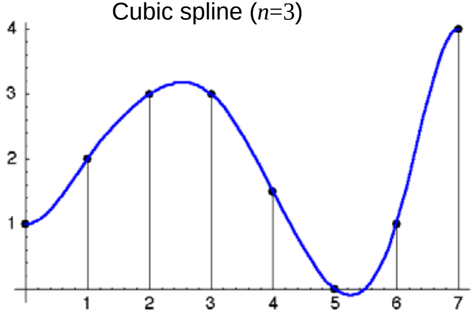
Polynomial splines (Schoenberg, 1946)

Definition:

$s(x)$ is *cardinal* polynomial spline of degree n iff

- Piecewise polynomial:
  $s(x)$ is a polynomial of degree n in each interval $[k, k+1)$;
- Higher-order continuity:
  $s(x), s^{(1)}(x), \ldots, s^{(n-1)}(x)$ are continuous at the knots k.

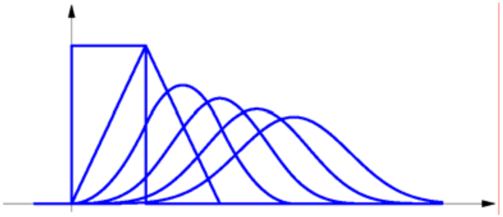
Cubic spline (n=3)



B-spline representation

- B-splines of degree *n*

$$\beta_+^n(x) = \underbrace{\beta_+^0 * \cdots * \beta_+^0}_{(n+1)\ \text{times}}(x)$$

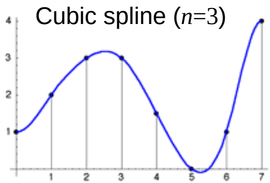


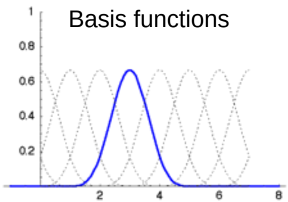
Explicit formula : $\beta_+^n(x) = \dfrac{\Delta_+^{n+1} x_+^n}{n!}$

Theorem [Schoenberg, 1946]

A cardinal spline of degree *n* has a stable, unique representation as a linear combination of shifted B-splines

$$s(x) = \sum_{k \in Z} c(k) \beta_+^n(x-k)$$





Can we fractionalize splines ?

- Schoenberg's formula

$$\beta_+^n(x) = \frac{\Delta_+^{n+1} x_+^n}{n!}$$

$$\downarrow$$

$$\beta_+^\alpha(x) = \frac{\Delta_+^{\alpha+1} x_+^\alpha}{\Gamma(\alpha+1)}$$

Basic tools for fractionalization

- Generalized factorials—Euler's Gamma function

$$n! = \Gamma(n+1) \qquad \Gamma(u) = \int_0^{+\infty} x^{u-1} e^{-x}\, dx$$

- Generalized binomial

$$(1+z)^{\gamma} = \sum_{k=0}^{+\infty} \binom{\gamma}{k} z^k \qquad \binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}$$

- Fractional derivative [Liouvillle, 1855]

$$\partial^s \xleftrightarrow{\;Fourier\;} (j\omega)^s$$

- Fractional finite differences

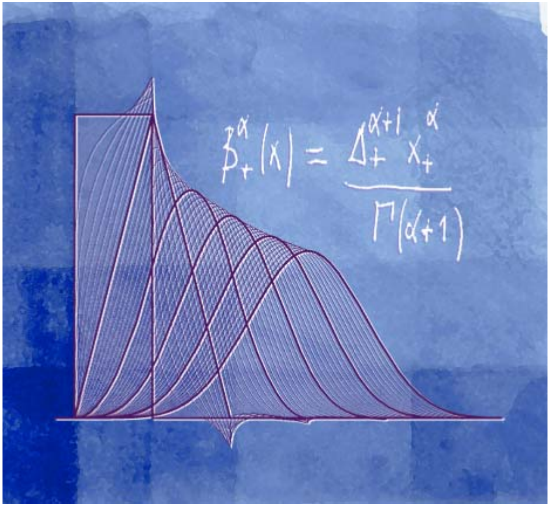
$$\Delta_+^s \xleftrightarrow{\;Fourier\;} (1-e^{-j\omega})^s \qquad \Rightarrow \qquad \Delta_+^s f(x) = \sum_{k=0}^{+\infty} (-1)^k \binom{s}{k} f(x-k)$$

Fractional B-splines

$$\beta_+^0(x) := x_+^0 - (x-1)_+^0 \quad \overset{Fourier}{\longleftrightarrow} \quad \left(\frac{1-e^{-j\omega}}{j\omega}\right)$$

$$\vdots$$

$$\beta_+^\alpha(x) := \frac{\Delta_+^{\alpha+1} x_+^\alpha}{\Gamma(\alpha+1)} \quad \overset{Fourier}{\longleftrightarrow} \quad \left(\frac{1-e^{-j\omega}}{j\omega}\right)^{\alpha+1}$$

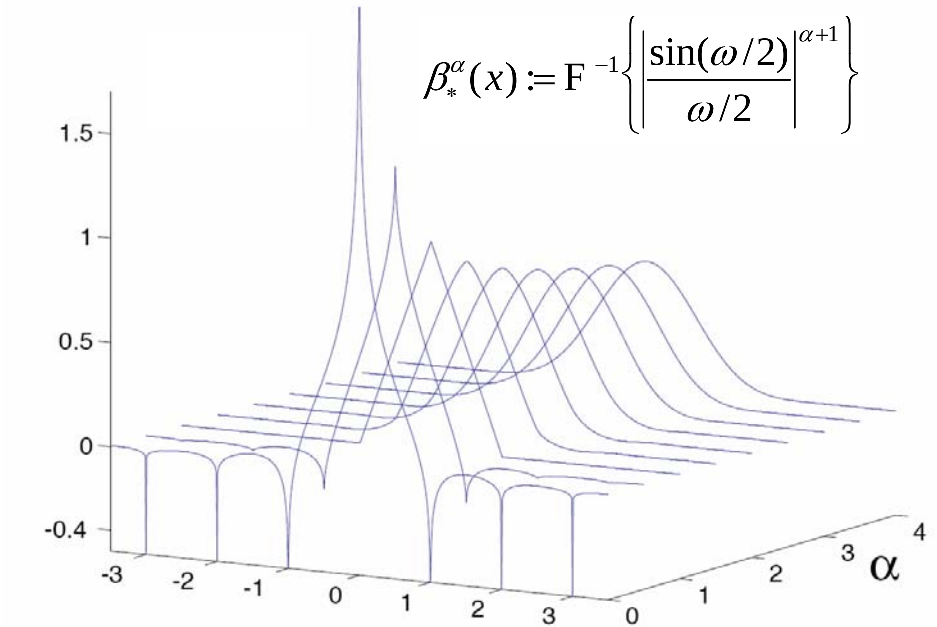


One-sided power functions: $x_+^\alpha = \begin{cases} x^\alpha & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

Symmetric B-splines

- Symmetrization in Fourier domain: $\left|\hat{\beta}_{+}^{\alpha}(x)\right| = \left|\left(\frac{1-e^{-j\omega}}{j\omega}\right)^{\alpha+1}\right| = \left|\frac{\sin(\omega/2)}{\omega/2}\right|^{\alpha+1}$

$$\beta_{*}^{\alpha}(x) \coloneqq \mathrm{F}^{-1}\left\{\left|\frac{\sin(\omega/2)}{\omega/2}\right|^{\alpha+1}\right\}$$



Properties

Generic notation : β^α for either β_+^α (causal) or β_*^α (symmetric)

- Equivalence with classical B-splines

  $\beta_+^\alpha(x)$ with $\alpha = n$ (integers)

  $\beta_*^\alpha(x)$ with $\alpha = 2n+1$ (odd integers)

  $\Rightarrow$ Compact support !

- Decay

  Theorem : For $\alpha > -1$, there exists a constant C such that $\left|\beta^\alpha(x)\right| \le \dfrac{C}{|x|^{\alpha+2}}$.

  (U. & Blu, *SIAM Rev*, 2000)

- Convolution property

  $$\beta^{\alpha_1} * \beta^{\alpha_2} = \beta^{\alpha_1+\alpha_2+1}$$

  $$\langle \beta^\alpha(\cdot), \beta^\alpha(\cdot - x)\rangle = \beta_*^{2\alpha+1}(x)$$

Riesz basis

$\{\beta^\alpha(x-k)\}_{k\in Z}$ is a Riesz basis for the cardinal fractional splines

- Generic B-spline representation of a fractional spline

$$s(x) = \sum_{k\in Z} c[k]\beta^\alpha(x-k) \quad \Leftrightarrow \quad \{c[k]\}_{k\in Z}$$

Continuous-time function (fractional spline)

Discrete representation (digital signal)

- Stable, one-to-one representation

For $\alpha > -\frac{1}{2}$, there exist two constants $A_\alpha > 0$ and $B_\alpha < +\infty$ such that

$$\forall c \in l_2, \qquad A_\alpha \cdot \|c\|_{l_2} \le \left\|\sum_{k\in Z} c[k]\beta^\alpha(x-k)\right\|_{L_2} \le B_\alpha \cdot \|c\|_{l_2}$$

Explicit fractional differentiation formula

- Fractional derivative operators

$$\partial^s \xleftrightarrow{\text{Fourier}} (j\omega)^s$$

$$\partial^s \beta_+^\alpha(x) = \Delta_+^s \beta_+^{\alpha-s}(x)$$

Fractional finite difference operator:

$$\Delta_+^s \xleftrightarrow{\text{Fourier}} (1-e^{-j\omega})^s$$

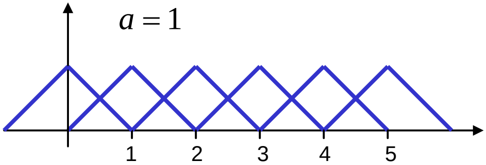
Sketch of proof:

$$\partial^s \beta_+^\alpha(x) \longleftrightarrow (j\omega)^s \cdot \left(\frac{1-e^{-j\omega}}{j\omega}\right)^{\alpha+1} = \left(1-e^{-j\omega}\right)^s \cdot \left(\frac{1-e^{-j\omega}}{j\omega}\right)^{\alpha+1-s}$$

Order of approximation

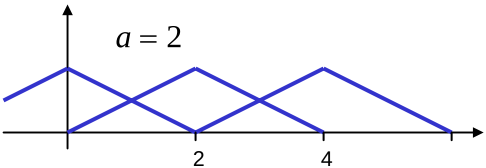
- Approximation space at scale a

$$V_a = \left\{ s_a(x) = \sum_{k \in Z} c(k)\varphi\left(\frac{x}{a} - k\right) : c(k) \in l_2 \right\}$$



- Projection operator

$$\forall f \in L_2, \qquad P_a f = \arg\min_{s_a \in V_a} \|f - s_a\|_{L_2} \qquad \in V_a$$



- Order of approximation

DEFINITION

A scaling function φ has order of approximation γ iff

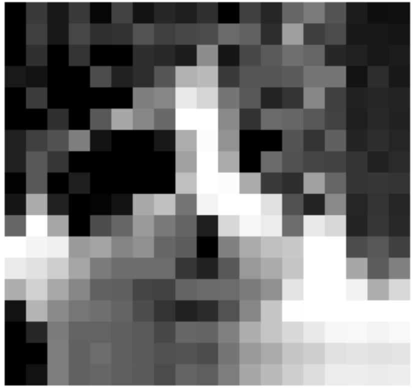
$$\forall f \in W_2^{\gamma}, \qquad \|f - P_a f\| \le C \cdot a^{\gamma} \left\| f^{(\gamma)} \right\| = O(a^{\gamma})$$

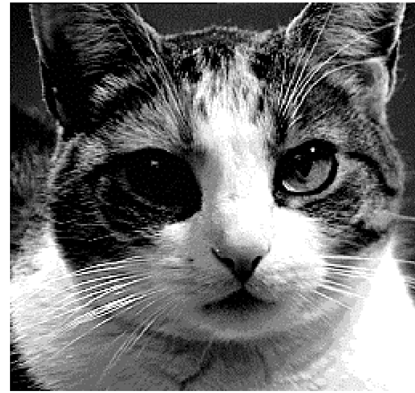
B-splines of degree α have order of approximation $\gamma = \alpha + 1$

Spline reconstruction of a CAT-scan

Piecewise constant

$\gamma = 1$





Cubic spline

$\gamma = 4$

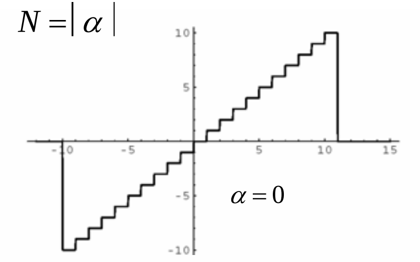
Reproduction of polynomials

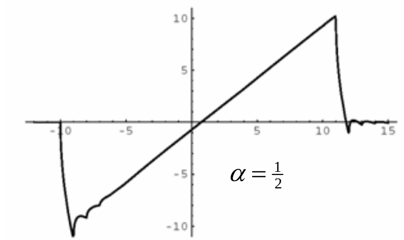
- B-splines reproduce polynomials of degree $N = \lceil \alpha \rceil$

$$\sum_{k \in Z} \beta_+^{\alpha}(x-k) = 1$$

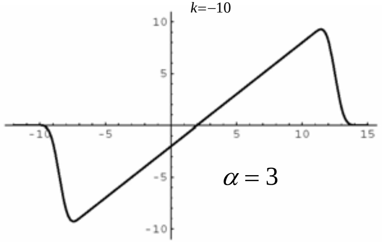
$$\vdots$$

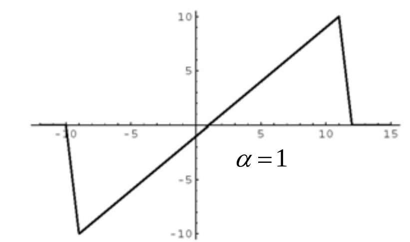
$$\sum_{k \in Z} k^n \beta_+^{\alpha}(x-k) = x^n + a_1 x^{n-1} + \cdots + a_n$$





$$\sum_{k=-10}^{+10} k \beta_+^{\alpha}(x-k)$$





More fractals…

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Pollock

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Dali

Mandelbrot meets Mondrian

FRACTIONAL WAVELETS

- Basic ingredients
- Constructing fractional wavelets
- Fractional B-spline wavelets
- Multi-scale fractional differentiation
- Adjustable wavelet properties

Scaling function

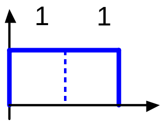
DEFINITION: $\varphi(x)$ *is an admissible scaling function of* L_2 *iff*:

- Riesz basis condition

$$\forall c \in l_2, \quad A \cdot \|c\|^2 \le \left\| \sum_k c(k)\varphi(x-k) \right\|_{L_2}^2 \le B \cdot \|c\|^2$$

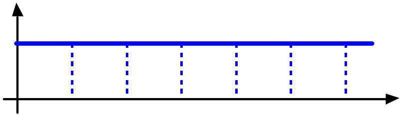
- Two-scale relation

$$\varphi(x/2) = 2\sum_{k \in Z} h(k)\varphi(x-k)$$



- Partition of unity

$$\sum_{k \in Z} \varphi(x-k) = 1$$



From scaling functions to wavelets

- Wavelet bases of L_2 (Mallat-Meyer, 1989)

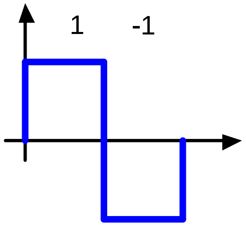
For any given admissible scaling function of L_2, $\varphi(x)$, *there exits a wavelet*

$$\psi(x/2) = 2\sum_{k\in Z} g(k)\varphi(x-k)$$

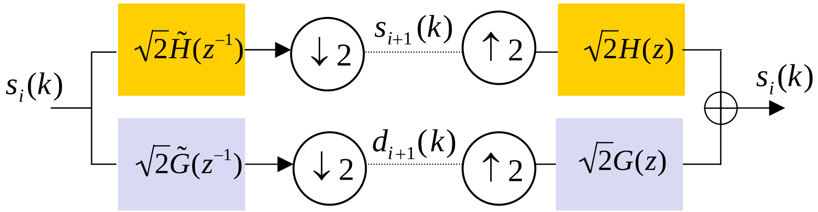
such that the family of functions

$$\left\{\frac{1}{\sqrt{2^i}}\psi\left(\frac{x-2^i k}{2^i}\right)\right\}_{i\in Z, k\in Z}$$

forms of Riesz basis of L_2.



- Constructive approach: perfect reconstruction filterbank



Constructing fractional wavelets

Theorem : Let $\varphi(x)$ be the L_2-stable solution (scaling function) of the two-scale relation

$$\varphi(x/2) = 2\sum_{k\in Z} h(k)\varphi(x-k)$$

Then $\varphi(x)$ is of order γ (**fractional**) if and only if

$$H(z) = \underbrace{\left(\frac{1+z^{-1}}{2}\right)^{\gamma}}_{\text{spline part}} \cdot \underbrace{Q(z)}_{\text{distributional part}} \quad \text{with} \quad \left|Q(e^{j\omega})\right| < \infty$$

$$\Updownarrow$$

Multi-scale differentiator		B-spline factorization:		Approximation order:
$\hat{\tilde{\psi}}(\omega) \propto (-j\omega)^{\gamma},\ \omega \to 0$	$\Leftrightarrow$	$\varphi = \beta_+^{\gamma-1} * \varphi_0$	$\Leftrightarrow$	$\left\|f - P_a f\right\|_{L_2} = O(a^{\gamma})$

$$\Downarrow$$

Vanishing moments:

$$\int x^n \tilde{\psi}(x)dx = 0,\ n = 0,\cdots\lceil\gamma-1\rceil$$

(Unser & Blu, *IEEE-SP*, 2003)

Binomial refinement filter

- Two-scale relation

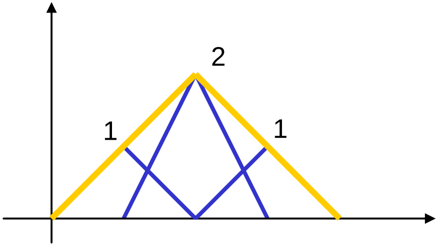
$$\beta_+^\alpha(x/2) = 2\sum_{k\in Z} h_+^\alpha(k)\beta_+^\alpha(x-k)$$

- Generalized binomial filter

$$h_+^\alpha(k) = \frac{1}{2^{\alpha+1}}\binom{\alpha+1}{k} \quad \longleftrightarrow \quad H^\alpha(z) = \left(\frac{1+z^{-1}}{2}\right)^{\alpha+1}$$

$$\binom{u}{v} = \frac{\Gamma(u+1)}{\Gamma(v+1)\Gamma(u-v+1)}$$

- Example of linear splines: α=1



Fractional B-spline wavelets

$$\psi_+^{\alpha}(x/2) = \sum_{k \in Z} \underbrace{\frac{(-1)^k}{2^{\alpha}} \sum_n \binom{\alpha+1}{n} \beta_*^{2\alpha+1}(n+k-1)}_{g(k)} \beta_+^{\alpha}(x-k)$$

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Video decompressor
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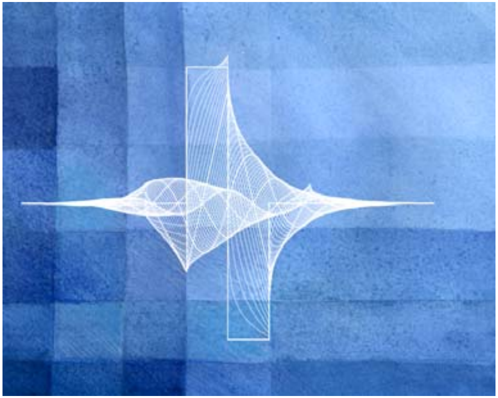
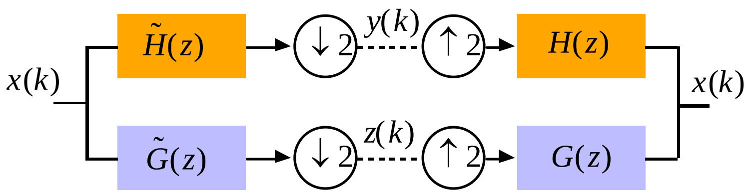


- **Remarkable property**

  Each of these wavelets generates a *semi-orthogonal* Riesz basis of L_2

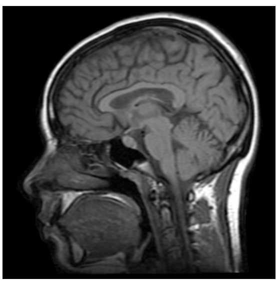
FFT-based wavelet algorithm

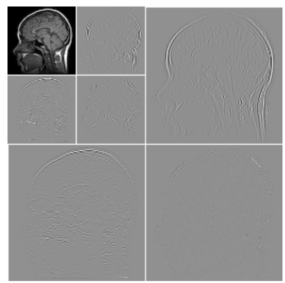
- Filterbank algorithm



$$\varphi(x/2) = \sqrt{2} \sum_{k \in Z} h(k) \varphi(x-k)$$

$$\psi(x/2) = \sqrt{2} \sum_{k \in Z} g(k) \varphi(x-k)$$

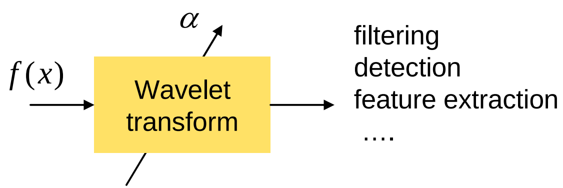




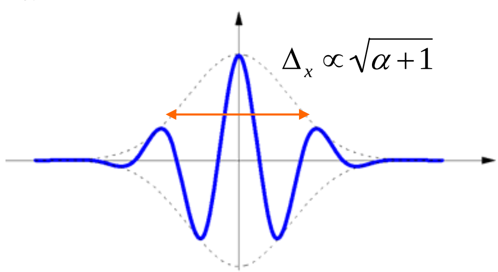
Click for demo

(Blu & Unser, *ICASSP*'2000)

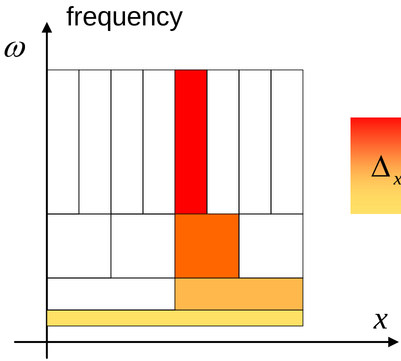
Adjustable wavelet properties



- Transform is tunable in a continuous fashion !
  - Order of differentiation: $\gamma=\alpha+1$
    - Whitening of fBMs, fractals …..
  - Regularity
    - Hölder continuity: α
    - Sobolev: $s_{max}= \alpha+1/2$
  - Localization:



Wavelets and the uncertainty principle



$$\Delta_x \cdot \Delta_\omega = \mathrm{Const}$$

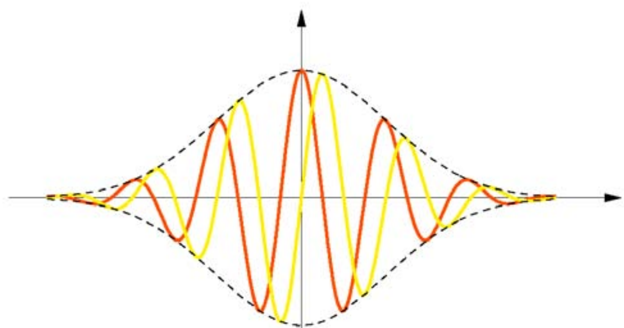
$$\Delta_x = \min_{x_0} \frac{\left\| (x - x_0)\psi(x) \right\|_{L_2}}{\left\| \psi \right\|_{L_2}}$$

$$\Delta_\omega = \min_{\omega_0} \frac{\left\| (\omega - \omega_0)\hat{\psi}(\omega) \right\|_{L_2}}{\left\| \hat{\psi} \right\|_{L_2}}$$

- Heisenberg's uncertainty relation

$$\boxed{\Delta_x \cdot \Delta_\omega \geq \frac{1}{2}}$$

with equality iff $\psi(x) = a \cdot e^{-b(x-x_0)^2 + j\omega_0 x}$



Question: are there such wavelet bases ?

Localization of the B-spline wavelets

Theorem

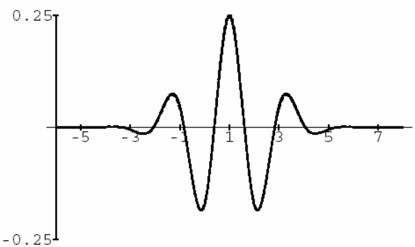
The B-spline wavelets converge (in L_p-norm) to modulated Gaussians as the degree goes to infinity :

$$\lim_{\alpha\to\infty}\left\{\beta_+^\alpha(x)\right\} = C\cdot e^{-(x-x_\alpha)^2/2\sigma_\alpha^2}$$

$$\lim_{\alpha\to\infty}\left\{\psi_+^\alpha(x)\right\} = \underbrace{C'\cdot e^{-(x-x'_\alpha)^2/2\sigma'^2_\alpha}}_{\text{Gaussian}} \times \underbrace{\cos(\omega_0 x + \theta_\alpha)}_{\text{sinusoid}}$$

$$\sigma_\alpha = \sqrt{\frac{\alpha+1}{12}}$$

$$\sigma'_\alpha = B\cdot\sigma_\alpha \quad \text{with} \quad B \cong 2.59$$



$\alpha = 3$

Cubic B-spline wavelets:
within 2% of the uncertainty limit !

(Unser et *al*., *IEEE-IT*, 1992)

QuickTime™ and a Video decompressor are needed to see this picture.

Are there wavelets in my brain ?

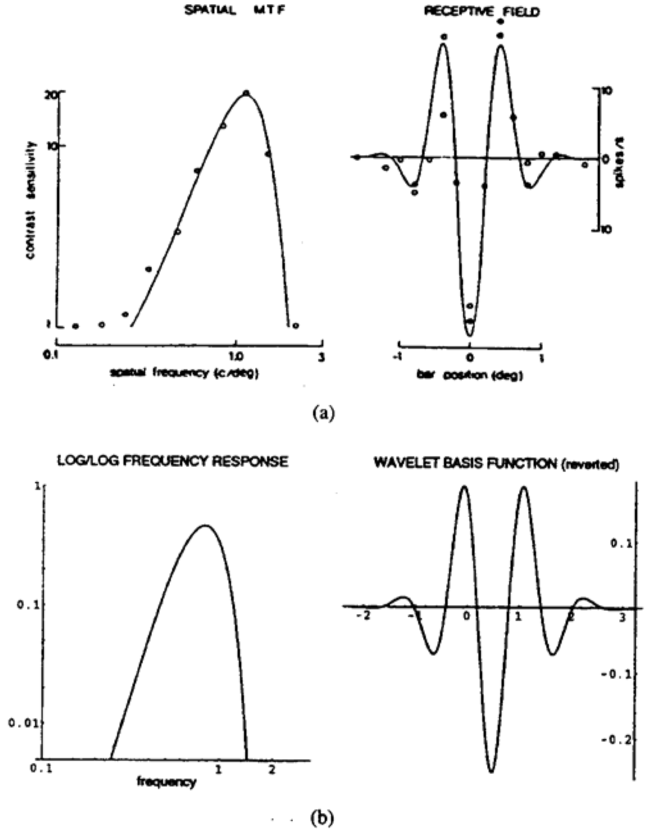
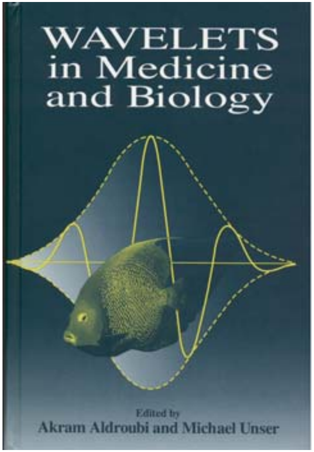


Fig. 2. Similarity between the receptive field of simple cortical cells and a wavelet basis function. (a) Response of a simple X cell from a monkey visual cortex and its fitted Gabor elementary signal [26], [67, Fig. 3]. (b) Semi-orthogonal cubic B-spline wavelet and its log-log frequency response [100].

WAVELETS IN MEDICAL IMAGING

- Survey of applications
- Analysis of functional imaging data (fMRI)



Wavelets in medical imaging: Survey 1991-1999

References

- Unser and Aldroubi, *Proc IEEE*, 1996
- Laine, *Annual Rev Biomed Eng*, 2000

- Special issue, *IEEE Trans Med Im*, 2003

Image processing task	Application / modality	Principal Authors
Image compression	• MRI • Mammograms • CT • Angiograms, etc…	Angelis 94; DeVore 95; Manduca 95; Wang 96; etc …
Filtering	*Image enhancement* • Digital radiograms • MRI • Mammograms • Lung X-rays, CT	Laine 94, 95; Lu, 94; Qian 95; Guang 97; etc …
	Denoising • MRI • Ultrasound (speckle) • SPECT	Weaver 91; Xu 94; Coifman 95; Abdel-Malek 97; Laine 98; Novak 98, 99
Feature extraction	*Detection of micro-calcifications* • Mammograms	Qian 95; Yoshida 94; Strickland 96; Dhawan 96; Baoyu 96; Heine 97, Wang 98
	Texture analysis and classification • Ultrasound • CT, MRI • Mammograms	Barman 93; Laine 94; Unser 95; Wei 95; Yung 95; Busch 97; Mojsilovic 97
	Snakes and active contours • Ultrasound	Chuang-Kuo 96
Wavelet encoding	• Magnetic resonance imaging	Weaver-Healy 92; Panych 94, 96; Geman 96; Shimizu 96; Jian 97
Image reconstruction	• Computer tomography • Limited angle data • Optical tomography • PET, SPECT	Olson 93, 94; Peyrin 94; Walnut 93; Delaney 95; Sahiner 96; Zhu 97; Kolaczyk 94; Raheja 99
Statistical data analysis	*Functional imaging* • PET • fMRI	Ruttimann 93, 94, 98; Unser 95; Feilner 99; Raz 99
Multi-scale Registration	*Motion correction* • fMRI, angiography *Multi-modality imaging* • CT, PET, MRI	Unser 93; Thévenaz 95, 98; Kybic 99
3D visualization	• CT, MRI	Gross 95, 97; Muraki 95; Kamath 98; Horbelt 99

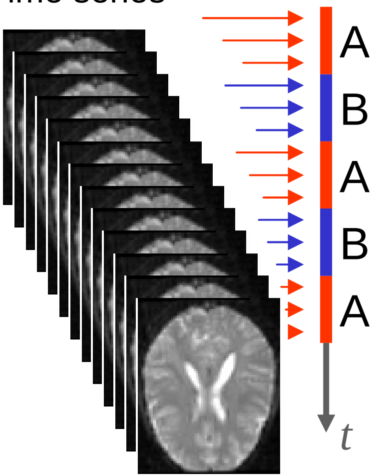
QuickTime™ and a TIFF (Uncompressed) decompressor are needed to see this picture.

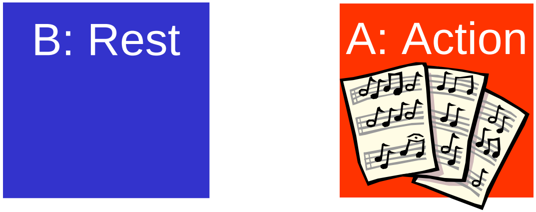
Functional brain imaging by fMRI

BOLD (Blood Oxygenation Level Dependence)

Basic principle: deoxygenated blood is more paramagnetic than oxygenated blood

Time series





EPI acquisition

Matrix size:	128 x 128 x 30 Pixels	x	68 measurements
Resolution:	1.56 x 1.56 x 4 mm	x	6 seconds

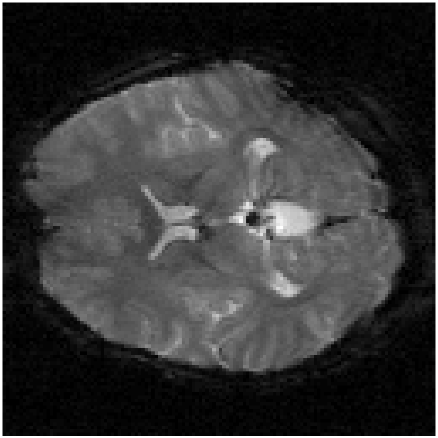
Functional brain imaging by fMRI (Cont'd)

Where?

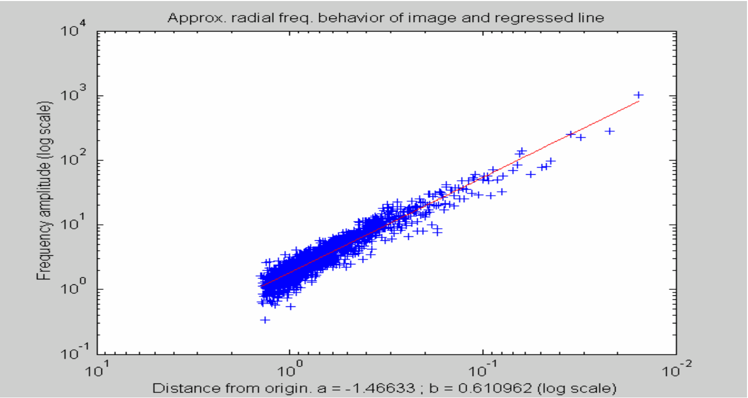
QuickTime™ and a decompressor are needed to see this picture.

- **Main problems**:
  - Small signal changes (1-5%)
  - Very noisy data — averaging
- **Standard solution**
  - Spatial Gaussian smoothing (SPM)

On the fractal nature of fMRI data



Brain: courtesy Jan Kybic

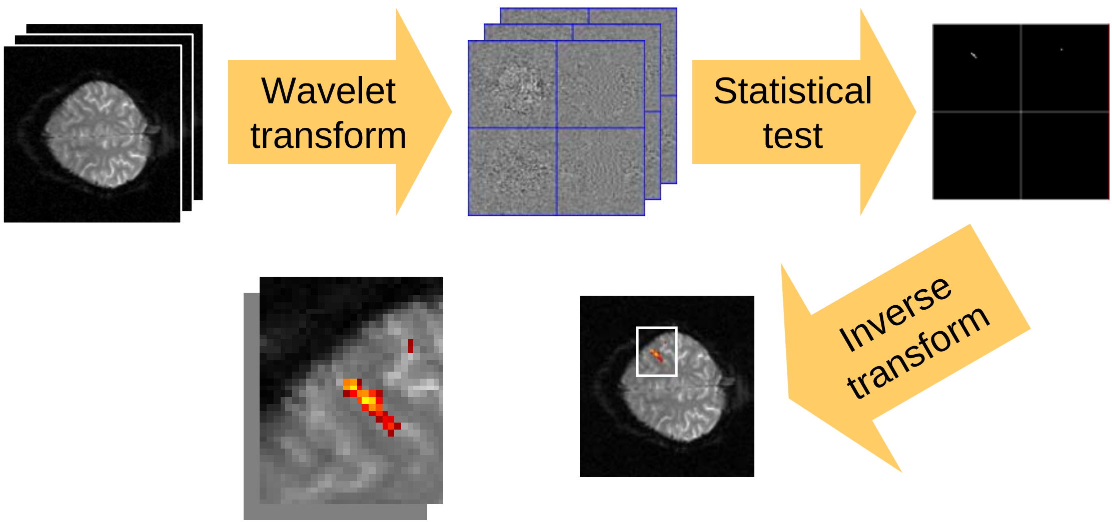


Log-Log plot of spectral density

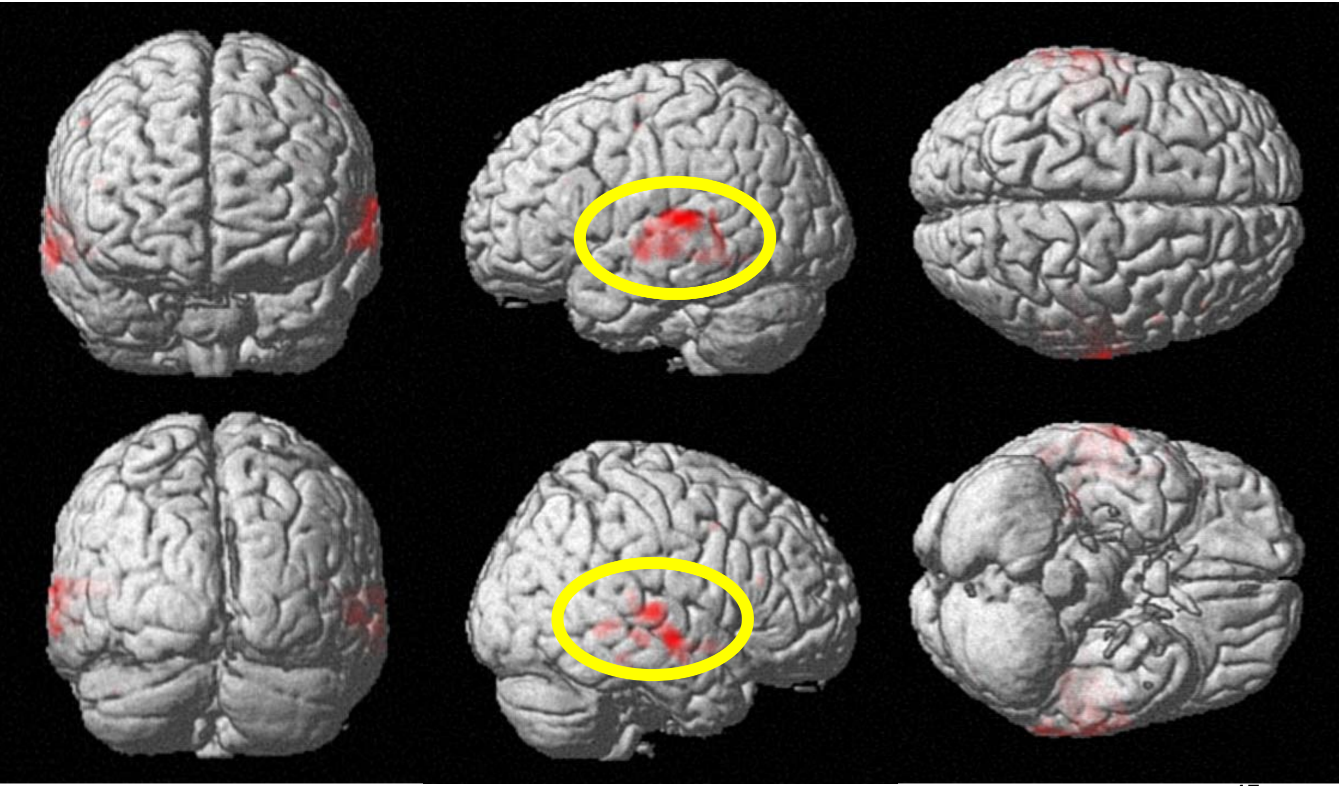
Fractal dimension: $D = 1 + d - H = 2.534 \quad \text{with} \quad d = 2$ (topological dimension)

Wavelet analysis of fMRI

- Advantages of the wavelet transform (Ruttiman et al., *IEEE-TMI*, 1998)
  - Orthogonal transformation : white noise $\rightarrow$ white noise
  - Decorrelates/whitens fMRI signal
  - Data compression
  - Increased signal-to-noise ratio (averaging effect)
  - Preserves space localization



An example: auditory stimulation



Conclusion

- Fractional splines
  - Natural extension of Schoenberg’s polynomial splines
  - Stable, convenient B-spline representation
  - Most polynomial B-spline properties are retained
  - Intimate link with fractional calculus
    - Elementary building blocks: Green functions of fractional derivative operators
  - Efficient digital-filter-based solutions
- New fractional wavelets
  - Multiresolution bases of L_2
  - Fast algorithm
  - Tunable
    - Regularity
    - Localization
    - Order of differentiation
  - Optimal for the processing of fractal-like processes (pre-whitening)
- Application in signal and image processing
  - Processing of fractal-like signals
  - Wavelet-based processing and feature extraction

Acknowledgments

Many thanks to

- Dr. Thierry Blu
- Annette Unser, Artist

\+ many other researchers, and graduate students

- Software and demos at: **http://bigwww.epfl.ch**

Extensions (on-going work)

- Richer family: alpha-tau splines

$$\partial_\tau^\gamma \xleftrightarrow{\text{Fourier}} (j\omega)^{\frac{\gamma}{2}+\tau}(-j\omega)^{\frac{\gamma}{2}-\tau}$$

[Blu et al., ICASSP'03]

- Multi-dimensional: fractional polyharmonic splines
  - Polyharmonic smoothing splines [Tirosh et al., ICASSP'04]
  - Polyharmonic wavelets [Van de Ville et al., under review]

$$\Delta^{\gamma/2} \xleftrightarrow{\text{Fourier}} \|\omega\|^\gamma$$